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An arithmetical equation with respect to regular convolutions

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Abstract

It is well known that Euler's totient function ϕ satisfies the arithmetical equation $\phi(mn)\phi((m,n)) = \phi(m)\phi(n)(m,n)$ for all positive integers m and n, where (m,n) denotes the greatest common divisor of m and n. In this paper we consider this equation in a more general setting by characterizing the arithmetical functions f with $f(1) \neq 0$ which satisfy the arithmetical equation $f(mn)f((m,n)) = f(m)f(n)g((m,n))$ for all positive integers m,n with $m,n \in A(mn)$, where A is a regular convolution and g is an A-multiplicative function. Euler's totient function ϕ_A with respect to A is an example satisfying this equation.

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1 Introduction

An arithmetical function f is said to be multiplicative if $f(1) = 1$ and $f(mn) = f(m)f(n)$ for all positive integers m,n with $(m,n) = 1$, where (m,n) denotes the greatest common divisor of m and n. A multiplicative function f is said to be completely multiplicative if $f(mn) = f(m)f(n)$ for all positive integers m,n. It is well known that Euler's totient function ϕ is multiplicative but not completely multiplicative. Thus ϕ is in a sense "between completely multiplicative and multiplicative functions". A well-known equation that reflects this property is given as

$$\phi(mn)\phi((m,n)) = \phi(m)\phi(n)(m,n) \tag{1}$$

for all positive integers m, n, see e.g. [2]. For all positive integers m, n with $(m,n) = 1$ this reduces to $\phi(mn) = \phi(m)\phi(n)$ showing that ϕ is multiplicative. Applying the formula

$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

we see that $\phi(mn) \neq \phi(m)\phi(n)$ for all positive integers m, n with $(m,n) > 1$.

An interesting question is to characterize the arithmetical functions satisfying an identity of the type of (1). We could characterize the arithmetical functions f with $f(1) = 1$ satisfying $f(mn)f((m,n)) = f(m)f(n)(m,n)$ for all m, n. It is however easy to consider a slightly more general problem, namely, to characterize the arithmetical functions f with $f(1) = 1$ satisfying

$$f(mn)f((m,n)) = f(m)f(n)g((m,n)) \tag{2}$$

for all positive integers m, n, where g is a completely multiplicative function. In fact, Apostol and Zuckerman [3] have shown that an arithmetical function f with $f(1) = 1$ satisfies (2) if and only if f is multiplicative and

$$f(p^{a+b})f(p^b) = f(p^a)f(p^b)g(p^b) \tag{3}$$

for all primes p and integers $a \geq b \geq 1$.

We obtain a more illustrative result if we assume that f possesses Property O which is defined as follows. We say that an arithmetical function f satisfies *Property* O if for each prime p, $f(p) = 0$ implies $f(p^a) = 0$ for all $a > 1$. Then (2) is a characterization of totients. An arithmetical function f is said to be a totient if there are completely multiplicative functions f_T and f_V such that

$$f = f_T * f_V^{-1}, \tag{4}$$

where $*$ denotes the Dirichlet convolution and f_V^{-1} is the inverse of f_V under the Dirichlet convolution. The functions f_T and f_V are referred to as the integral and inverse part of f. Now we are in a position to present the promised characterization of totients (given in [8]). An arithmetical function f is a totient if and only if f with $f(1) = 1$ satisfies Property O and there exists a completely multiplicative function g such that (2) holds. In this case $f_T = g$.

It is well known that Euler's totient function ϕ can be written as

$$\phi = N * \mu = N * \zeta^{-1},$$

where $N(n) = n$ and $\zeta(n) = 1$ for all positive integers n and μ is the Möbius function. Thus ϕ is a totient in the sense of (4) with $\phi_T = N$ and $\phi_V = \zeta$. In this case (2) reduces to (1). Any arithmetical function f with $f(1) = 1$ and Property O satisfying $f(mn)f((m,n)) = f(m)f(n)(m,n)$ for all positive integers m and n is a totient with $f_T = N$, that is, $f = N * f_V^{-1}$. Dedekind's totient ψ defined as $\psi = N * |\mu|$ is another totient with this property, since $|\mu| = \lambda^{-1}$, where $|\mu|(n) = |\mu(n)|$ is the absolute value of the the Möbius function and λ is Liouville's function, which the completely multiplicative function such that $\lambda(p) = -1$ for all primes p. Thus Dedekind's totient ψ satisfies the arithmetical equation

$$\psi(mn)\psi((m,n)) = \psi(m)\psi(n)(m,n) \tag{5}$$

for all positive integers m, n.

In this paper we investigate the arithmetical equation (2) in a more general setting. Namely, we introduce a generalization of (2) for regular convolutions. Suppose that A is a regular convolution and g is an A-multiplicative arithmetical function (defined in Section 2). Then we consider the arithmetical functions f with $f(1) \neq 0$ which satisfy the arithmetical equation

$$f(mn)f((m,n)) = f(m)f(n)g((m,n)) \tag{6}$$

for all positive integers m, n with $m, n \in A(mn)$. (Note that the condition $m, n \in A(mn)$ is equivalent to the condition $m \in A(mn)$.)

In the case of the Dirichlet convolution the arithmetical equation (6) becomes (2). Equation (6) has not hitherto been studied in the literature. Equation (2) has been studied in [1, 3, 6, 8]. For further material relating to this type of equations we refer to [4, 13]. Theorems 1, 3, 4 of this paper are generalizations of Theorems 2, 3 and 4 of [3]. Theorem 2 generalizes the main theorem of [6], and Corollary 2 generalizes Theorem 3 of [1] (see also Lemma 4.1 of [5]). Corollary 1 generalizes Theorem 10 of [8] and shows that the functional equation (6) is closely related to totient type functions.

2 Preliminaries

For each positive integer n let $A(n)$ be a subset of the set of positive divisors of n. Then the A-convolution [11] of two arithmetical functions f and g is defined by

$$(f *_A g)(n) = \sum_{d \in A(n)} f(d) g(n/d).$$

An A-convolution is said to be regular [11] if

(i) the set of arithmetical functions forms a commutative ring with identity with respect to the usual addition and the A-convolution,

(ii) the multiplicativity of f and g implies the multiplicativity of $f *_A g$,

(iii) the function ζ has an inverse μ_A with respect to the A-convolution, and $\mu_A(n) = 0$ or -1 whenever n $(\neq 1)$ is a prime power.

The inverse of an arithmetical function f with $f(1) \neq 0$ with respect to an A-convolution satisfying condition (i) is defined by

$$f^{-1} *_A f = f *_A f^{-1} = \delta,$$

where δ is the arithmetical function such that $\delta(1) = 1$ and $\delta(n) = 0$ for $n > 1$.

It is known [11] that an A-convolution is regular if and only if

(a) $A(mn) = \{de : d \in A(m), d \in A(n)\}$ whenever $(m, n) = 1$,

(b) for each prime power p^a with $a > 0$ there exists a positive integer t $(= \tau_A(p^a))$ such that

$$A(p^a) = \{1, p^t, p^{2t}, \ldots, p^{st}\},$$

where $st = a$ and

$$A(p^{it}) = \{1, p^t, p^{2t}, \ldots, p^{it}\},\ 0 \leq i \leq s.$$

For example the Dirichlet convolution D, defined by $D(n) = \{d > 0 : d|n\}$, and the unitary convolution U, defined by $U(n) = \{d > 0 : d|n, (d, n/d) = 1\}$, are regular convolutions. We assume throughout this paper that A is an arbitrary but fixed regular convolution.

A positive integer n is said to be A-primitive if $A(n) = \{1, n\}$. For each A-primitive prime power p^t $(\neq 1)$ the order [7] of p^t is defined by

$$o(p^t) = \sup\{s : \tau_A(p^{st}) = t\}.$$

For the Dirichlet convolution D the primes are the only D-primitive prime powers and the order of each prime is infinity. For the unitary convolution

U all prime powers are U-primitive and the order of each prime power ($\neq 1$) is equal to 1.

In this paper we often write $1 \leq i \leq o(p^t)$. Here i is an integer, and if $o(p^t) = \infty$, we adopt the convention that $1 \leq i \leq o(p^t)$ means $1 \leq i$ (that is, i is a positive integer).

An arithmetical function f is said to be quasi-A-multiplicative [7] if $f(1) \neq 0$ and

$$f(1)f(mn) = f(m)f(n) \text{ whenever } m, n \in A(mn).$$

It can be shown that a quasimultiplicative function f is quasi-A-multiplicative if and only if

$$f(p^{it}) = f(1)^{1-i} f(p^t)^i, \quad 1 \leq i \leq o(p^t), \tag{7}$$

where p^t ($\neq 1$) is an A-primitive prime power. Thus a quasi-A-multiplicative function is totally determined by its values at A-primitive prime powers.

Quasi-A-multiplicative functions f with $f(1) = 1$ are said to be A-multiplicative functions [15]. An arithmetical function f with $f(1) \neq 0$ is quasi-A-multiplicative if and only if $f/f(1)$ is A-multiplicative. Quasi-U-multiplicative functions are known as quasimultiplicative functions [9], U-multiplicative functions are the usual multiplicative functions, and D-multiplicative functions are the usual completely multiplicative functions. Therefore, for example, quasimultiplicative functions are defined by the conditions

$$\begin{aligned} &f(1) \neq 0, \\ &f(1)f(mn) = f(m)f(n) \text{ whenever } (m, n) = 1. \end{aligned}$$

An arithmetical function f is said to be a quasi-A-totient [7] if

$$f = f_T *_A f_V^{-1},$$

where f_T and f_V are quasi-A-multiplicative functions. The inverse of a quasi-A-multiplicative function g is given as

$$g^{-1} = \frac{\mu_A g}{g(1)^2},$$

see [7]; thus a quasi-A-totient f can be written in the form

$$f = f_T *_A \left(\frac{\mu_A f_V}{f_V(1)^2} \right).$$

The generalized Möbius function μ_A is the multiplicative given by

$$\mu_A(p^a) = \begin{cases} 1 & \text{if } a = 0, \\ -1 & \text{if } p^a \ (\neq 1) \text{ is } A\text{-primitive}, \\ 0 & \text{otherwise.} \end{cases}$$

An arithmetical function f with $f(1) \neq 0$ is a quasi-A-totient if and only if $f/f(1)$ is an A-totient. It is easy to see that D-totients are the usual totients and U-totients are simply the usual multiplicative functions.

The function $\phi_A(n)$ is defined as the number of integers $a \pmod n$ such that $(a,n)_A = 1$, where $(a,n)_A$ is the grestest divisor of a that belongs to $A(n)$. It is well known [10] that $\phi_A = N *_A \mu_A$, and therefore ϕ_A is an A-totient with $f_T = N$ and $f_V = \zeta$.

Dedekind's totient function ψ_A with respect to a regular convolution is defined as $\psi_A = N *_A |\mu_A|$. Let λ_A denote Liouville's function with respect to a regular convolution, that is, λ_A is the A-multiplicative function such that $\lambda(p^t) = -1$ for all A-primitive prime powers p^t $(\neq 1)$. Then $\psi_A = N *_A \lambda_A^{-1}$, and therefore ψ_A is the A-totient with $f_T = N$ and $f_V = \lambda_A$.

For general accounts on regular convolutions and related arithmetical functions we refer to [10, 12, 14].

3 Characterization

We characterize the arithmetical functions f with $f(1) \neq 0$ satisfying (6). We assume that g is an A-multiplicative function in (6). Then $g(1) = 1$.

Theorem 1. *An arithmetical function f with $f(1) \neq 0$ satisfies (6) if and only if f is quasimultiplicative and*

$$f(p^{(a+b)t})f(p^{bt}) = f(p^{at})f(p^{bt})g(p^{bt}) \tag{8}$$

for all A-primitive prime powers p^t $(\neq 1)$ and $a \geq b \geq 1$, $a + b \leq o(p^t)$.

Proof. Assume that (6) holds. Then taking $(m,n) = 1$ in (6) gives $f(mn)f(1) = f(m)f(n)$, and therefore f is quasimultiplicative. Furthermore, taking $m = p^{at}$, $n = p^{bt}$ with $a + b \leq o(p^t)$ (where p^t $(\neq 1)$ is an A-primitive prime power) in (6) proves (8).

Conversely, assume that f is a quasimultiplicative function satisfying (8). We show that (6) holds. Since f is quasimultiplicative, it is enough to show that (6) holds when m and n are prime powers. If $m = 1$ or $n = 1$, then (6) holds. If $m \neq 1$ and $n \neq 1$, then there is an A-primitive prime power p^t $(\neq 1)$

such that $m = p^{at}$, $n = p^{bt}$ with $2 \le a+b \le o(p^t)$, since $m, n \in A(mn)$. Then (6) reduces to (8), and therefore, by assumption, (6) holds. This completes the proof.

Making a further assumption on f we see that (6) is closely related to totient type functions.

Property O_A. We say that an arithmetical function f satisfies Property O_A if for each A-primitive prime power p^t ($\neq 1$), $f(p^t) = 0$ implies $f(p^{at}) = 0$ for all $1 \le a \le o(p^t)$.

By virtue of (7), all quasi-A-multiplicative functions possess Property O_A. Since $\phi_A(p^t) = p^t - 1 \neq 0$ and $\psi_A(p^t) = p^t + 1 \neq 0$, the functions ϕ_A and ψ_A possess Property O_A.

Theorem 2. *An arithmetical function f with $f(1) \neq 0$ is a solution of* (6) *with Property O_A if and only if f is quasimultiplicative and there is an A-multiplicative function g such that*

$$f(p^{at}) = f(p^t)g(p^t)^{a-1} \tag{9}$$

for all A-primitive powers p^t and $1 \le a \le o(p^t)$.

Proof. Assume that (6) and Property O_A hold. Then (8) holds by Theorem 2. Taking $b = 1$ in (8) we obtain

$$f(p^{(a+1)t})f(p^t) = f(p^{at})f(p^t)g(p^t),$$

where $2 \le a+1 \le o(p^t)$. In this way we see that

$$\begin{aligned} f(p^{(a+1)t})f(p^t) &= f(p^{at})f(p^t)g(p^t) = f(p^{(a-1)t})f(p^t)g(p^t)^2 = \cdots \\ &= f(p^t)f(p^t)g(p^t)^a, \end{aligned}$$

where $2 \le a+1 \le o(p^t)$. Thus

$$f(p^{at})f(p^t) = f(p^t)f(p^t)g(p^t)^{a-1}, \tag{10}$$

where $2 \le a \le o(p^t)$. Clearly (10) holds even for $1 \le a \le o(p^t)$. If $f(p^t) \neq 0$, then (10) reduces to (9). If $f(p^t) = 0$, then (9) holds by Property O_A.

Conversely, assume that f is a quasimultiplicative function satisfying (9). Let $a \ge b \ge 1$, $a+b \le o(p^t)$. Applying (9) we obtain

$$\begin{aligned} f(p^{(a+b)t})f(p^{bt}) &= f(p^t)g(p^t)^{a+b-1}f(p^t)g(p^t)^{b-1} \\ &= f(p^t)g(p^t)^{a-1}f(p^t)g(p^t)^{b-1}g(p^{bt}) \\ &= f(p^{at})f(p^{bt})g(p^{bt}), \end{aligned}$$

which shows that (8) holds. Thus, by Theorem 1, f is a solution of (6). On the basis of (9) we see that Property O_A holds. This completes the proof.

Lemma 1 ([7]). *Suppose that f is a quasi-A-totient. Then*

$$f(p^{at}) = \left(\frac{f_T(p^t)}{f_T(1)}\right)^{a-1} f(p^t),\ 1 \le a \le o(p^t),$$

for all A-primitive prime powers p^t ($\neq 1$).

Conversely, suppose that f is quasimultiplicative and for all A-primitive prime powers p^t ($\neq 1$) there exists a complex number $z(p^t)$ such that

$$f(p^{at}) = (z(p^t))^{a-1} f(p^t),\ 1 \le a \le o(p^t).$$

Then f is a quasi-A-totient with $f_T(p^t)/f_T(1) = z(p^t)$.

Corollary 1. *Suppose that f is a quasi-A-totient. Then*

$$f(mn)f((m,n))f_T(1) = f(m)f(n)f_T((m,n)) \tag{11}$$

whenever $m, n \in A(mn)$.

Conversely, suppose that there exists a quasi-A-multiplicative function g such that

$$f(mn)f((m,n))g(1) = f(m)f(n)g((m,n)) \tag{12}$$

whenever $m, n \in A(mn)$, and f satisfies Property O_A. Then f is a quasi-A-totient with

$$\frac{f_T(p^t)}{f_T(1)} = \frac{g(p^t)}{g(1)}.$$

Proof. Suppose that f is a quasi-A-totient. Then f is quasimultiplicative. By Lemma 1, f satisfies (9) with $g(p^t) = f_T(p^t)/f_T(1)$. Thus, by Theorem 2, f satisfies (6) with $g = f_T/f_T(1)$. This means that (11) holds.

Conversely, suppose that f satisfies (12) and Property O_A. Then f satisfies (6) with g replaced with $g/g(1)$, which is A-multiplicative. Thus, by Theorem 2 and Lemma 1, is a quasi-A-totient with $f_T(p^t)/f_T(1) = g(p^t)/g(1)$.

Example 1. Euler's totient function ϕ_A with respect to a regular convolution satisfies the arithmetical equation

$$\phi_A(mn)\phi_A((m,n)) = \phi_A(m)\phi_A(n)(m,n) \tag{13}$$

whenever $m, n \in A(mn)$, and Dedekind's totient function ψ_A with respect to a regular convolution satisfies the same arithmetical equation

$$\psi_A(mn)\psi_A((m,n)) = \psi_A(m)\psi_A(n)(m,n) \tag{14}$$

whenever $m, n \in A(mn)$.

Corollary 2. *Suppose that g is a quasi-A-multiplicative function and h is a multiplicative function with*

$$g(p^t)[g(p^t) - h(p^t)] \neq 0$$

for all A-primitive prime powers p^t $(\neq 1)$. Denote $f = g *_A (\mu_A h)$. Then

$$f(mn) = f(m)f(n)\frac{g((m,n))}{f((m,n))g(1)}$$

whenever $m, n \in A(mn)$.

Proof. Let u be the A-multiplicative function such that $u(p^t) = h(p^t)$ for all A-primitive prime powers p^t $(\neq 1)$. Then $u^{-1} = \mu_A u = \mu_A h$, and thus $f = g *_A u^{-1}$, which means that f is a quasi-A-totient with $f_T = g$. This implies that $f(1) \neq 0$. Further, since $f(p^{at}) = g(p^t)^{a-1}[g(p^t) - h(p^t)] \neq 0$ for all A-primitive prime powers p^t $(\neq 1)$ and $1 \leq a \leq o(p^t)$ and since f is multiplicative, f is always nonzero. Now, the claim follows from Corollary 1.

Next we examine (8) without assuming Property O_A. We distinguish two cases: $g(p^t) = 0$, $g(p^t) \neq 0$.

Theorem 3. *Let p^t $(\neq 1)$ be an A-primitive prime power, and let g be an A-multiplicative function with $g(p^t) = 0$. Then an arithmetical function f satisfies* (8) *for all $a \geq b \geq 1$ and $a + b \leq o(p^t)$ if and only if there exists an integer c (depending on p^t) with $1 \leq c \leq o(p^t)$ such that*

$$f(p^{it}) = 0 \text{ for } 1 \leq i \leq c - 1 \text{ and } 2c \leq i \leq o(p^t). \tag{15}$$

Proof. Since $g(p^t) = 0$, then (8) becomes

$$f(p^{(a+b)t})f(p^{bt}) = 0. \tag{16}$$

Now, assume that (15) holds. We show that (16) holds. Choose two integers a, b such that $a \geq b \geq 1$ and $a + b \leq o(p^t)$. If $b \leq c - 1$, then $f(p^{bt}) = 0$ and consequently (16) holds. If $b \geq c$, then $a + b \geq 2b \geq 2c$ and thus $f(p^{(a+b)t}) = 0$. Therefore (16) holds. So we have proved that (16) or (8) holds.

Conversely, suppose that (8) holds, that is, (16) holds. We show that (15) holds. If $f(p^{it}) = 0$ whenever $1 \leq i \leq o(p^t)$, then (15) holds with $c = 1$. Assume then that $f(p^{it}) \neq 0$ for some $1 \leq i \leq o(p^t)$. Let c be the smallest i for which $f(p^{it}) \neq 0$ and $1 \leq i \leq o(p^t)$. Then $f(p^{ct}) \neq 0$ and $f(p^{it}) = 0$ for $i \leq c - 1$. Next, suppose $2c \leq i \leq o(p^t)$. Write $i = a + c$ $(a \geq c)$. Taking $b = c$ in (16) proves that $f(p^{it}) = f(p^{(a+b)t}) = 0$. So we have proved that (15) holds. This completes the proof.

Example 2. Let g be the A-multiplicative function with $g(p^t) = 0$ for all A-primitive prime powers p^t $(\neq 1)$. Then $g = \delta$ and (6) becomes

$$f(mn)f((m,n)) = f(m)f(n)\delta((m,n)) \tag{17}$$

for $m, n \in A(mn)$, which means that $f(mn)f(1) = f(m)f(n)$ when $(m,n) = 1$ and $f(mn)f((m,n)) = 0$ when $(m,n) > 1$ and $m, n \in A(mn)$. Now, let $c = 1$ for all A-primitive prime powers p^t $(\neq 1)$ in Theorem 3. Then $f(p^{it}) = 0$ for $2 \leq i \leq o(p^t)$, which means that $f = \mu_A h$ for some quasimultiplicative function h. Thus, the function $f = \mu_A h$ possesses the property (17). For instance, the function $f = \mu_A$ possesses the property (17).

Theorem 4. *Let p^t $(\neq 1)$ be an A-primitive prime power, and let g be an A-multiplicative function with $g(p^t) \neq 0$. Then an arithmetical function f satisfies* (8) *if and only if there exists an arithmetical function h (which may depend on p^t) such that*

$$f(p^{at}) = h(a)g(p^t)^a \textit{ for all } 1 \leq a \leq o(p^t), \tag{18}$$

where h satisfies the functional equation

$$h(a+b)h(b) = h(a)h(b) \textit{ for all } a \geq b \geq 1, a+b \leq o(p^t). \tag{19}$$

Proof. Assume that there exists an arithmetical function h satisfying (19), and let $f(p^{at})$ be given by (18). Then, for $a \geq b \geq 1$, $a + b \leq o(p^t)$ we have

$$f(p^{(a+b)t})f(p^{bt}) = h(a+b)g(p^t)^{a+b}h(b)g(p^t)^b$$

and

$$f(p^{at})f(p^{bt})g(p^{bt}) = h(a)g(p^t)^a h(b)g(p^t)^b g(p^{bt}).$$

Since g is A-multiplicative, $g(p^{bt}) = g(p^t)^b$. Now, by (19), we obtain (8).

Conversely, suppose that (8) holds. Take $h(a) = f(p^{at})/g(p^t)^a$ for $1 \leq a \leq o(p^t)$. Then, $f(p^{at}) = h(a)g(p^t)^a$, and thus (8) becomes

$$h(a+b)g(p^t)^{a+b}h(b)g(p^t)^b = h(a)g(p^t)^a h(b)g(p^t)^b g(p^{bt}).$$

Since $g(p^{bt}) = g(p^t)^b$ and $g(p^t) \neq 0$, we obtain (19). This completes the proof.

Example 3. Suppose that f is a quasi-A-totient and p^t $(\neq 1)$ is an A-primitive prime power. Then, by Lemma 1,

$$f(p^{at}) = f(p^t)\left(\frac{f_T(p^t)}{f_T(1)}\right)^{a-1}, \ 1 \leq a \leq o(p^t).$$

If $f_T(p^t) \neq 0$, then

$$f(p^{at}) = \frac{f_T(1)f(p^t)}{f_T(p^t)} \left(\frac{f_T(p^t)}{f_T(1)}\right)^a = h(a)g(p^t)^a,$$

where $h(a) = f_T(1)f(p^t)/f_T(p^t)$ (a constant) for all $1 \leq a \leq o(p^t)$ and $g(p^t) = f_T(p^t)/f_T(1)$. It is clear that h satisfies (19) and $f_T/f_T(1)$ is an A-multiplicative function. Thus f satisfies (8).

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